Strain Gradient Elasticity and Dual Internal Variables

Arkadi Berezovski

Abstract The framework of dual internal variables approach is used to formulate a strain gradient elasticity model. The elimination of internal variables results in an equation of motion with no causality issues. Furthermore, the Aifantis-type strain gradient model is reproduced if the primary internal variable is proportional to the Laplacian of the strain gradient.

1 Introduction

The strain gradient elasticity is a natural extension of classical theory that takes into account the internal length scale (Aifantis, 2016). Classical linear elasticity theory is characterized by the Hooke law which in one-dimensional case takes the form of the stress-strain relation

$$\sigma = \rho c^2 \varepsilon, \tag{1}$$

where σ is the stress, ρ is the density, ε is the strain, c is the elastic wave speed. The corresponding equation of motion in terms of the displacement u is the well known wave equation

$$u_{tt} = c^2 u_{xx}. \tag{2}$$

Accordingly, the free energy density W is a quadratic function of the strain

$$W = \frac{\rho c^2}{2} \varepsilon^2. \tag{3}$$

Taking the strain gradient into account, the quadratic free energy is represented as

Arkadi Berezovski
Department of Cybernetics, School of Science, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia, e-mail: arkadi.berezovski@cs.ioc.ee

$$W = \frac{\rho c^2}{2}\varepsilon^2 + A\varepsilon\varepsilon_x + \frac{1}{2}B(\varepsilon_x)^2. \tag{4}$$

Then the stress is generalized, by definition,

$$\sigma = \frac{\partial W}{\partial \varepsilon} = \rho c^2 \varepsilon + A\varepsilon_x, \tag{5}$$

and the double stress is, correspondingly,

$$\mu = \frac{\partial W}{\partial \varepsilon_x} = A\varepsilon + B\varepsilon_x. \tag{6}$$

In terms of stresses, the equation of motion takes the form (Mindlin and Eshel, 1968)

$$\rho u_{tt} = \sigma_x - \mu_{xx}, \tag{7}$$

which, in terms of displacement, is transformed to

$$\rho u_{tt} = \rho c^2 u_{xx} - B u_{xxxx}. \tag{8}$$

It is instructive to visualize dispersion properties of this equation. Considering harmonic wave with the frequency ω and wavenumber k

$$u(x,t) = \hat{u} e^{i(kx-\omega t)}, \tag{9}$$

we arrive at the dispersion relation

$$\rho \omega^2 = \rho c^2 k^2 + B k^4. \tag{10}$$

In terms of dimensionless frequency and wavenumber

$$\omega' = \frac{\omega}{\omega_0}, \quad k' = \frac{ck}{\omega_0}. \tag{11}$$

we have

$$\omega'^2 = k'^2 + B' k'^4, \tag{12}$$

where ω_0 is a characteristic frequency and $B' = B\omega_0^2/\rho c^4$. The corresponding dispersion curve is shown in Fig. 1. As can be seen, only acoustical branch of the dispersion curve is reproduced (Lombardo and Askes, 2012; Madeo et al., 2013, c.f.). The dispersion curve with positive B is unrealistic (Askes et al., 2002; Metrikine and Askes, 2002).

The same situation holds true for a low frequency, long wavelength approximation of the micromorphic elasticity (Mindlin, 1964) with vanishing relative deformation (Mindlin and Eshel, 1968; Fleck and Hutchinson, 1997; Lam et al., 2003; Zervos, 2008; Dell'Isola et al., 2009; Forest and Sab, 2012; Rosi et al., 2018) based on the free energy density for linear isotropic elasticity in the form (Mindlin and Eshel, 1968)

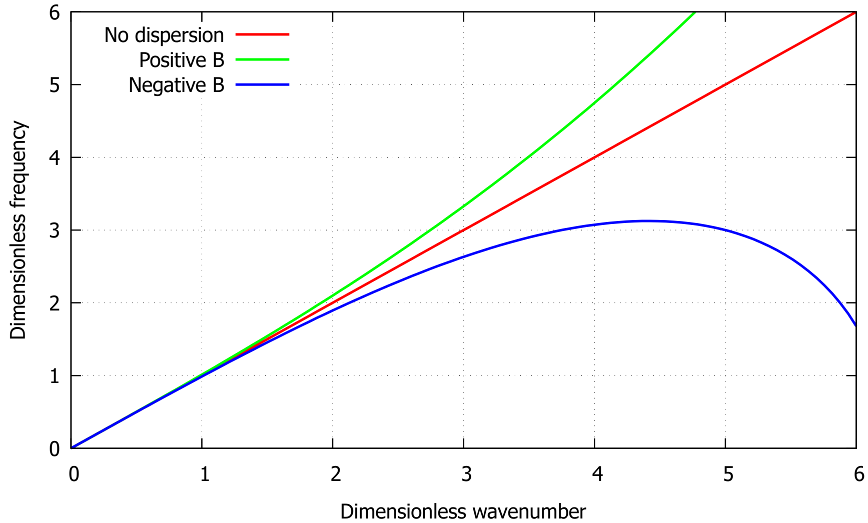

Fig. 1 Dispersion curves for B' = 0.0256.

$$W = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\varepsilon_{ik,i}\varepsilon_{jj,k} + a_2\varepsilon_{jj,i}\varepsilon_{kk,i} + \\ + a_3\varepsilon_{ik,i}\varepsilon_{jk,j} + a_4\varepsilon_{jk,i}\varepsilon_{jk,i} + a_5\varepsilon_{jk,i}\varepsilon_{ij,k}, \tag{13}$$

where ε_{ij} is the strain tensor, λ and μ are Lamé constants, a_i are additional constitutive parameters.

Another approach to strain gradient elasticity is represented by a Laplacian-based theory of gradient elasticity provided by Aifantis and coworkers (Aifantis, 1992; Ru and Aifantis, 1993; Altan and Aifantis, 1997)

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - l^2(\lambda\delta_{ij}\varepsilon_{kk,mm} + 2\mu\varepsilon_{ij,mm}), \tag{14}$$

where σ_{ij} is the stress tensor and l is a length scale parameter.

This idea, which has been extensively elaborated by Askes and coworkers (Askes et al., 2002; Metrikine and Askes, 2002; Askes and Metrikine, 2002; Askes and Aifantis, 2006; Askes et al., 2008; Askes and Gitman, 2010; Askes and Aifantis, 2011; DeDomenico and Askes, 2016) as well as by other researchers (Polizzotto, 2014; Broese et al., 2016; Polizzotto, 2017; Khakalo and Niiranen, 2018; Forest, 2020; Broese et al., 2021), is unable to reveal the optical branch of the dispersion curve due to the absence of higher order time derivatives in the equations of motion. This is related to the problem of causality (Metrikine, 2006) and the possibility of infinite wave propagation speed. The causality problem can be solved by either phenomenologically adding the higher order time derivatives (Metrikine, 2006; Askes et al., 2008) or by using internal variables (Engelbrecht et al., 2005; Berezovski et al., 2009, 2011; Engelbrecht and Berezovski, 2015; Berezovski et al., 2020).

The goal of this paper is to show how dual internal variables can be used to model strain gradient elasticity while avoiding the causality problem. Numerous strain gradient applications (Askes and Aifantis, 2011; Aifantis, 2020) necessitate this.

2 Dual internal variables

In the small-strain approximation, the local balance laws governing the motion of elastic solids are (no body forces)

$$\frac{\partial(\rho v_i)}{\partial t} - \sigma_{ij,j} = 0, \tag{15}$$

$$\frac{\partial \mathcal{E}}{\partial t} - \sigma_{ij}\dot{\varepsilon}_{ij} + Q_{i,i} = 0, \tag{16}$$

and the second law of thermodynamics takes the following form:

$$\frac{\partial S}{\partial t} + \left(\frac{Q_i}{\theta}\right)_{,i} \geq 0. \tag{17}$$

Here v_i is the particle velocity, $\mathcal{E}$ is the internal energy per unit volume, S is the entropy density, Q_i is the heat flux, θ is temperature, dot over a symbol denotes time derivative.

Applying the dual internal variable concept (Ván et al., 2008), we consider internal variables as second-order tensors. In this approach, the free energy per unit volume W is represented as a sufficiently regular function of strain, temperature, and two unspecified internal variables φ_{ij} and ψ_{ij}

$$W = \overline{W}(\varepsilon_{ij}, \theta, \varphi_{ij}, \psi_{ij}). \tag{18}$$

The stress and entropy are defined as usual

$$\sigma_{ij} = \frac{\partial \overline{W}}{\partial \varepsilon_{ij}}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \tag{19}$$

and two quantities conjugated to internal variables are introduced as follows:

$$A_{ij} := -\frac{\partial \overline{W}}{\partial \varphi_{ij}}, \quad B_{ij} := -\frac{\partial \overline{W}}{\partial \psi_{ij}}. \tag{20}$$

In terms of the free energy $W = \mathcal{E} - \theta S$, entropy inequality (17) is transformed to

$$-\frac{\partial W}{\partial t} - S\frac{\partial \theta}{\partial t} + \left(\frac{Q_i}{\theta}\right)\theta_{,i} + \sigma_{ij}\dot{\varepsilon}_{ij} \geq 0. \tag{21}$$

The time derivative of the free energy is determined by accepted functional dependence (18) and equations of state (19)

$$\begin{aligned}\frac{\partial \overline{W}}{\partial t} =&\frac{\partial \overline{W}}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial t}+\frac{\partial \overline{W}}{\partial \theta}\frac{\partial \theta}{\partial t}+\frac{\partial \overline{W}}{\partial \varphi_{ij}}\frac{\partial \varphi_{ij}}{\partial t}+\frac{\partial \overline{W}}{\partial \psi_{ij}}\frac{\partial \psi_{ij}}{\partial t}=\\ =&\sigma_{ij}\dot{\varepsilon}_{ij}-S\dot{\theta}-A_{ij}\dot{\varphi}_{ij}-B_{ij}\dot{\psi}_{ij}.\end{aligned} \tag{22}$$

Here dot over a symbol denotes time derivative.

When we combine relationship (22) and dissipation inequality (21) we get

$$-\frac{Q_i}{\theta}\theta_{,i}+A_{ij}\dot{\varphi}_{ij}+B_{ij}\dot{\psi}_{ij}\geq 0. \tag{23}$$

Consequently, dissipation is governed by thermal and internal factors.

2.1 Evolution equations

In the isothermal case, the dissipation inequality is expressed in terms of internal variables

$$A_{ij}\dot{\varphi}_{ij}+B_{ij}\dot{\psi}_{ij}\geq 0. \tag{24}$$

If the case is non-dissipative, the most straightforward solution of Eq. (24) is

$$\dot{\varphi}_{ij}=B_{ij}, \tag{25}$$

$$\dot{\psi}_{ij}=-A_{ij}. \tag{26}$$

In this case, the evolution of one internal variable is driven by another, indicating the duality of the internal variables.

2.2 Quadratic free energy

A quadratic function of the arguments is used to represent the free energy

$$W=\frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+D_1\varepsilon_{ij}\varphi_{ij}+\frac{D_2}{2}\varphi_{ij}\varphi_{ij}+\frac{D_3}{2}\psi_{ij}\psi_{ij}. \tag{27}$$

The internal variable φ_{ij} is regarded as primary while the internal variable ψ_{ij} is considered auxiliary. Equations of state that correspond to them are

$$\sigma_{ij} = \frac{\partial \overline{W}}{\partial \varepsilon_{ij}} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} + D_1 \varphi_{ij},$$
$$A_{ij} = -\frac{\partial \overline{W}}{\partial \varphi_{ij}} = -D_1 \varepsilon_{ij} - D_2 \varphi_{ij}, \quad B_{ij} = -\frac{\partial \overline{W}}{\partial \psi_{ij}} = -D_3 \psi_{ij}. \tag{28}$$

In terms of the displacement, the balance of linear momentum takes the form

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,ij} + \mu u_{i,jj} + D_1 \varphi_{ij,j}. \tag{29}$$

The evolution equation for the internal variable φ is determined by elimination the second internal variable using equations (25) and (26)

$$\ddot{\varphi}_{ij} = -D_3 \dot{\psi}_{ij} = D_3 A_{ij} = -D_3 D_1 \varepsilon_{ij} - D_3 D_2 \varphi_{ij}. \tag{30}$$

Furthermore, we can write a single equation of motion in terms of displacement differentiating equation of motion (29) twice with respect to time and applying evolution equation (30)

$$\rho \ddot{\ddot{u}}_i - (\lambda+\mu) \ddot{u}_{j,ij} - \mu \ddot{u}_{i,jj} = -D_3 D_1^2 \frac{1}{2}(u_{i,jj} + u_{j,ij}) - D_3 D_2 (\rho \ddot{u}_i - (\lambda+\mu) u_{j,ij} - \mu u_{i,jj}). \tag{31}$$

We expect that the internal variable φ_{ij} is a function of the strain gradient, i.e., $\varphi_{ij} = \varphi(\varepsilon_{ij,k})$. That is, the second rank tensor φ_{ij} must be expressed in terms of the third rank tensor $\varepsilon_{ij,k}$. The obvious choice for such a relationship is the proportionality of internal variable φ_{ij} to the Laplacian of the strain gradient

$$\varphi_{ij} = k \varepsilon_{ij,kk} = k \Delta \varepsilon_{ij}, \tag{32}$$

where k is an appropriate constant. The application of Eq. (32) in the definition of the stress $(28)_1$ results in the Aifantis strain gradient model (Aifantis, 1992; Ru and Aifantis, 1993; Altan and Aifantis, 1997; Askes et al., 2002; Metrikine and Askes, 2002; Askes and Aifantis, 2011)

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} + k D_1 \Delta \varepsilon_{ij}. \tag{33}$$

However, equation of motion (31) is free from the causality problem.

3 Concluding remarks

The use of dual internal variables allows to eliminate the causality problems in the corresponding dispersive wave equation in the context of strain gradient elasticity. The Aifantis model for the strain gradient elasticity is reproduced by proportionality of the internal variable to the Laplacian of the strain tensor. Since the third-order strain gradient tensor is not explicitly included in the presented model, it does not replace existing strain gradient models. The model can be combined with other strain gradient models that are currently available (Lam et al., 2003; Askes and Aifantis, 2006, 2011; Bacigalupo and Gambarotta, 2014; DeDomenico and Askes, 2016; Zhou et al., 2016; Madeo et al., 2017; Polizzotto, 2017; Khakalo and Niiranen, 2018; DeDomenico et al., 2019; Fu et al., 2020).

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